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# Vibration of a viscous liquid sphere

H H K Tang† and C Y Wong‡

†Department of Physics, Kalamazoo College, Kalamazoo, Michigan 49001, USA
‡Oak Ridge National Laboratory, Oak Ridge, Tennessee 37830, USA

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**Abstract.** Previously, a characteristic equation for the vibration of a viscous gravitational globe had been obtained by Chandrasekhar. The same equation has been found to apply for a viscous liquid drop under the restoring force of surface tension and is proved here to apply to all combinations of gravitational attraction, surface tension and Coulomb repulsion with surface or volume charge distribution. Complex solutions to this Chandrasekhar equation corresponding to the periodic motion with damping, which have not been found up to now but are called for in a large class of physical problems, are calculated. In addition, some solutions for higher aperiodic modes of decay are also evaluated.

## 1. Introduction

The vibration of a spherical liquid mass was first considered by Lord Kelvin (1863) who obtained the eigenfrequencies for the vibrations. The corresponding problem of a liquid drop was solved by Lord Rayleigh (1882). Chandrasekhar (1959) investigated the effect of viscosity in the case of an incompressible self-gravitating mass. A characteristic equation was obtained for the eigenfrequencies which were in general complex numbers. The corresponding problem of a viscous uncharged liquid drop was solved by Reid (1960) who established that except for a difference in the fundamental frequencies due to the different nature of forces involved, the two cases were formally identical and the same Chandrasekhar characteristic equation applied.

Surface tension and gravitational forces are not the only ones that may be present in a spherical globe. One can envisage the presence of electric charge and consider any combination of the three types of forces in a globe. While surface tension or gravitational force can act alone as the restoring force for the liquid, in the case of a charged liquid drop, the Coulomb repulsion must be counterbalanced by a strong enough surface tension. It is of interest to generalize Chandrasekhar's and Reid's results to the case where the vibrating droplet is endowed with electric charge, distributed uniformly either over the volume or on the surface. Our analysis indicates that irrespective of the combination of forces, the same Chandrasekhar equation applies. Henceforth, we shall use the term 'liquid sphere' for the general case to denote a gravitating globe or a charged or uncharged liquid drop, as the case may be.

Because of the general nature of the Chandrasekhar characteristic equation, its solutions may find applications to many physical problems in which vibrations of fluid drops are involved. For example, the vibrations of a water drop are of meteorological interest. Recently, extensive experimental and theoretical work has been done on vibrating droplets under various circumstances. The oscillations of atmospheric water drops provide a mechanism for drop break-up within the clouds and are directly related to the formation of rain (Billings and Holland 1969, Ausman and Brook 1967, Nelson and Gokhale 1972). It has also been discovered (Brook and Latham 1968) that vibrating water drops in the atmosphere have a back-scattering effect on radar. From the fluctuations of the radar return signal thus obtained, size distribution of rain drops can be estimated. The vibrations of electrified drops and their break-up in electric fields have been found to play a central role in the formation of thunderstorms (Azad and Latham 1970, Rosenkilde 1969, Brazier-Smith *et al* 1971, Brazier-Smith 1971, Jennings and Latham 1971). Recently, a new technique has been developed for the electromechanical break-up of liquid drops (Hendriks and Babli 1972), with a view to applications in fuel injection in controlled thermonuclear reactors.

On the other hand, a self-gravitating sphere of viscous liquid may be of astrophysical interest, as it can serve as a first approximate model of a star whose non-radial oscillation may be induced if initially deformed by some external force. The related problem of the vibration of a spinning and self-gravitating viscous spheroid may also be a good idealization of astrophysical objects endowed with large angular momentum (Lebovitz 1961, Rosenkilde 1967, Chandrasekhar 1969). Another case of interest is that of an atomic nucleus where a model of charged liquid drop has approximate validity (Bohr and Wheeler 1939) It is hoped that an understanding of the classical motion of a viscous charged drop may provide some help for the problem of quantization later on.

It is clear that the vibration of these objects will be affected by the presence of viscosity. Although the Chandrasekhar equation has been known for a long time, complex solutions to the equation, corresponding to periodic vibrational motion with damping, are still lacking. Since the characteristic equation determines the properties of the motion of the liquid sphere, we study this equation in detail. We evaluate the complex solutions which give the oscillatory modes and are of most physical interest. Previously, some real solutions corresponding to the lowest aperiodic modes of the liquid drop were given. Additional real solutions corresponding to higher modes of decay are also calculated in this paper.

## 2. Governing equations and solutions

The governing equation for the internal motion of the fluid is the Navier–Stokes equation

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = -\left[\frac{\nabla p}{\rho_M} + \left(\frac{\rho_Q}{\rho_M}\right)\nabla V + \nu \,\mathrm{curl}^2 \boldsymbol{u}\right], \tag{1}$$

where $\boldsymbol{u}$ is the velocity characterizing the motion of the fluid element, $\rho_M$ is the mass density, $p$ is the pressure, $\nu = \eta/\rho_M$ is the kinematic viscosity, $V$ is the non-local gravitational or electrostatic potential, and $\rho_Q$ is the corresponding mass or charge density, as the case may be.

Following Chandrasekhar, we linearize the Navier–Stokes equation by assuming that the amplitude of vibration $\epsilon$ is small. The boundary of the mass is considered to be sharp and can be described by

$$r(\theta, \phi, t) = R + \epsilon(t) Y_{lm}(\theta, \phi), \tag{2}$$

where $R$ is the radius of the liquid sphere in equilibrium and $Y_{lm}(\theta, \phi)$ is a spherical harmonic. We further assume that the mass is incompressible and the density is uniform. Thus from the equation of continuity, we have div $\boldsymbol{u} = 0$. This condition, together with the solenoidal nature of the restoring force vector leads to the important implication that $\boldsymbol{u}$ is purely poloidal (Chandrasekhar 1968) (which means that $\boldsymbol{u}$ can be written in the form $\boldsymbol{u} = \nabla \times [\nabla \times (\Phi/r)\boldsymbol{r}]$, where $\Phi$ is a scalar function). From this, the mathematical analysis can be considerably simplified. In spherical coordinates, the various components of $\boldsymbol{u}$ (for the poloidal solution) are:

$$u_r = \mathrm{e}^{-\sigma t}\frac{l(l+1)}{r^2}U(r)Y_{lm}(\theta, \phi) \tag{3a}$$

$$u_\theta = \mathrm{e}^{-\sigma t}\frac{1}{r}\frac{\mathrm{d}U(r)}{\mathrm{d}r}\frac{\partial Y_{lm}(\theta, \phi)}{\partial \theta} \tag{3b}$$

$$u_\phi = \mathrm{e}^{-\sigma t}\frac{1}{r\sin\theta}\frac{\mathrm{d}U(r)}{\mathrm{d}r}\frac{\partial Y_{lm}}{\partial \phi} \tag{3c}$$

where $\sigma$ gives the frequency or decay rate of the motion and $U(r)$ is some function to be determined by the boundary conditions.

The Chandrasekhar characteristic equation has the same form for a viscous globe under gravitation or a liquid drop under surface tension (Chandrasekhar 1959, Reid 1959)

$$\Psi(z) \equiv -z^4 + 2(l-1)(2l+1)z^2 - 4(l-1)^2(l+1)z^2\frac{Q_{l+\frac{1}{2}}(z)}{z - 2Q_{l+\frac{1}{2}}(z)} = \alpha^4 \tag{4}$$

where

$$z \equiv \left(\frac{\sigma R^2}{\nu}\right)^{1/2} \equiv \alpha\left(\frac{\sigma}{\sigma_{l,0}}\right)^{1/2}, \tag{4a}$$

$$Q_{l+\frac{1}{2}}(z) \equiv \frac{J_{l+\frac{3}{2}}(z)}{J_{l+\frac{1}{2}}(z)} \tag{4b}$$

and

$$\alpha^2 \equiv \frac{\sigma_{l,0}R^2}{\nu}. \tag{4c}$$

The $J$ are Bessel functions of the first kind and $\sigma_{l,0}$ is the fundamental frequency in the inviscid case. Introduced this way, the quantity $\alpha^2$ is a generalized dimensionless viscosity coefficient.

## 3. Inviscid case

In the absence of viscosity, the natural frequencies $\sigma_{l,0}$ are different for the gravitating globe and the liquid drop under surface tension. For the gravitating globe, we have the Kelvin mode:

$$\sigma_{l,0}^2 = \tfrac{8}{3}\pi G\rho_M\frac{l(l-1)}{2l+1}, \tag{5a}$$

and for the liquid drop, we have the Rayleigh mode:

$$\sigma_{l,0}^2 = \frac{l(l-1)(l+2)T}{\rho_M R^3}, \tag{5b}$$

where $G$ is the gravitational constant and $T$ is the coefficient of surface tension.

In the case of a uniformly charged liquid drop, it can be shown that the fundamental frequency (Bohr and Wheeler mode) is given as

$$\sigma_{l,0}^2 = \frac{l(l-1)(l+2)}{\rho_M R^3}\left(T - \frac{3}{2}\frac{1}{(l+2)(2l+1)}\frac{kQ^2}{\pi R^3}\right), \tag{5c}$$

where $k$ is the coupling constant of the Coulomb interaction (it equals $(\frac{1}{4}\pi\epsilon_0)^{-1}$ in the usual notation) and $Q$ is the total excess charge endowed on the liquid drop.

When the charge is distributed only over the surface of the liquid drop, two different cases present themselves.

(i) When the liquid is a very good conducting material, the boundary can be taken as an equipotential surface during the oscillatory motion. We then have a fundamental frequency due to Rayleigh (1882) (see also Hendriks and Schneider 1962)

$$\sigma_{l,0}^2 = \frac{l(l-1)(l+2)}{\rho_M R^3}\left(T - \frac{kQ^2}{4\pi(l+2)R^3}\right). \tag{5d}$$

(ii) On the other hand, when the conductivity is very low, the charge cannot redistribute in time to form an equipotential surface during oscillation. We then have a different frequency (appendix)

$$\sigma_{l,0}^2 = \frac{l(l-1)(l+2)}{\rho_M R^3}T - \frac{l(l^2-3l-2)}{2l+1}\frac{kQ^2}{4\pi\rho_M R^6}. \tag{5e}$$

## 4. Viscous charged liquid drop under surface tension

It is desirable to consider the deviations in pressure and potential energy per unit mass in the perturbed state. Denoting the corresponding quantities by $V_0$ and $p_0$ at equilibrium, the deviations are:

$$\delta V(\boldsymbol{r}, t) \equiv V(\boldsymbol{r}, t) - V_0(\boldsymbol{r})$$

$$\delta p(\boldsymbol{r}, t) \equiv p(\boldsymbol{r}, t) - p_0(\boldsymbol{r}).$$

We define

$$\delta w = \frac{\delta p}{\rho_M} + \left(\frac{\rho_Q}{\rho_M}\right)\delta V, \tag{6}$$

then the linearized Navier–Stokes equation can be written as

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla\,\delta w - \nu\,\mathrm{curl}^2\boldsymbol{u}, \tag{7}$$

where terms proportional to $u^2$ are neglected. Taking the divergence of this equation and solving the resulting Laplace equation for interior points, we have the electric potential

$$\delta V = f(l)kQ\left(\frac{r^l}{R^{l+2}}\right)\epsilon Y_{lm}$$

where $f(l)$ is a coefficient depending on $l$ and the type of charge distribution.

(i) For the case where the drop is uniformly charged over the volume, we have

$$f(l) = \frac{3}{2l+1}.$$

(ii) For the case where the charge is only on the liquid surface and where the liquid has a very high conductivity, we have

$$f(l) = 0.$$

(iii) For surface charge distribution on a liquid with very low conductivity, the function $f(l)$ is (see appendix)

$$f(l) = \frac{-(l+1)}{2l+1}.$$

At the boundary points, the deviation in pressure is given as

$$\left(\frac{\delta p}{\rho_M}\right)_{R+\epsilon Y_{lm}} = (l-1)(l+2)\frac{T}{\rho_M R^2}\epsilon Y_{lm} - B(l)\epsilon Y_{lm}, \tag{8}$$

where $T$ is the coefficient of surface tension and $B(l)$ is a coefficient dependent on $l$ and on the charge distribution:

$$B(l) = \begin{cases} \dfrac{3}{4}\dfrac{kQ^2}{\pi\rho_M R^5} & \text{for uniform volume charge,} \\ (l-1)\dfrac{kQ^2}{4\pi\rho_M R^5} & \text{for surface charge with high conductivity,} \\ \dfrac{l^2-3l-2}{2l+1}\dfrac{kQ^2}{4\pi\rho_M R^5} & \text{for surface charge with low conductivity.} \end{cases}$$

From equation (6) and the Laplace equation, $\delta w$ has the form

$$\delta w = (l+1)\Pi_0 r^l \epsilon Y_{lm}, \tag{9}$$

where $\Pi_0$ is a constant to be determined from (6) and (7).

The following differential equation is obtained by combining equations (7), (3*a*), (3*b*) and (3*c*):

$$\frac{d^2U(r)}{dr^2} - \frac{l(l+1)}{r^2}U(r) + \frac{\sigma}{\nu}U(r) = \frac{\epsilon_0}{\nu}\Pi_0 r^{l+1}, \tag{10}$$

where we have written $\epsilon_0 e^{-\sigma t}$ in place of $\epsilon$. The general solution to this equation is

$$U(r) = Ar^{1/2}J_{l+\frac{1}{2}}(qr) + \frac{\epsilon_0}{\sigma}\Pi_0 r^{l+1} \tag{11}$$

where $q = (\sigma/\nu)^{1/2}$ and $A$ is a constant.

There are three boundary conditions to be considered. First, from the requirement of consistency between the radial component of the velocity and the form of the boundary, we have

$$l(l+1)\left(\frac{AJ_{l+\frac{1}{2}}(z)}{R^{3/2}}+\frac{\epsilon_0}{\sigma}\Pi_0 R^{l-1}\right) = -\epsilon_0\sigma. \tag{12}$$

Secondly, the tangential viscous stress tensor vanishes on the boundary. This leads to

$$\frac{\mathrm{d}^2U}{\mathrm{d}r^2}-\frac{2}{r}\frac{\mathrm{d}U}{\mathrm{d}r}+\frac{l(l+1)}{r^2}U = 0 \qquad \text{for } r = R. \tag{13}$$

Equation (13) together with equation (11), gives

$$A = \frac{2(l-1)\epsilon_0\sigma R^{3/2}}{l(2zJ_{l+\frac{1}{2}}(z)-z^2J_{l+\frac{1}{2}}(z))}.$$

One notes in passing that for small values of the viscosity, we have $A \sim \nu$ and the solution of equation (11) for $A = 0$ leads to

$$\nabla\times\boldsymbol{u} = 0.$$

In other words, one has an irrotational flow for the inviscid case. The flow becomes rotational as $A$ becomes non-zero. The effect of viscosity on flow pattern is therefore to change it from an irrotational flow into a rotational flow.

Finally, the last boundary condition stipulates that the $r$–$r$ component of the total stress tensor is given by $T(1/R_1+1/R_2)$, where $R_1$ and $R_2$ are the principal radii of curvature. The latter quantity can be shown to be (cf Lamb 1931, Landau and Lifschitz 1958)

$$T\left(\frac{1}{R_1}+\frac{1}{R_2}\right) = T\left(\frac{2}{R}+(l-1)(l+2)\frac{\epsilon Y_{lm}}{R^2}\right). \tag{14}$$

On the other hand, the $r$–$r$ component of the stress tensor is

$$-P_{rr} = p_0+\delta p-2\nu\rho_M\frac{\partial u_r}{\partial r}. \tag{15}$$

Thus, from equations (6), (11), (14), and (15), we arrive at the following equation:

$$\epsilon Y_{lm}\left(\Pi_0(l+1)R^l-\frac{R}{l}\sigma_{l,0}^2\right)-2\nu\left(\frac{\partial u_r}{\partial r}\right)_R = 0. \tag{16}$$

This equation is in the same form as equation (37) in the paper of Chandrasekhar (1959) for the case of a self-gravitating liquid sphere. Hence, it is clear that equation (16) will eventually lead to the same Chandrasekhar characteristic equation (4).

For completeness, one can consider the rather academic case of a gravitating sphere with surface tension. It can be proved that one obtains the same Chandrasekhar equation, with only the modification that the natural frequencies are:

$$\sigma_{l,0}^2 = l(l-1)(l+2)\frac{T}{\rho_M R^3}+\tfrac{8}{3}\pi G\rho_M\frac{l(l-1)}{2l+1}. \tag{17}$$

Thus, the Chandrasekhar equation is found to be applicable to all combinations of gravitational interaction, surface tension and Coulomb repulsion. Solutions to this equation can therefore be of interest for a very large class of physical problems.

## 5. Analysis and numerical results for aperiodic motion

We discuss below some properties of the Chandrasekhar characteristic equation (4) for the aperiodic modes, which have been given by Chandrasekhar (1959).

For real arguments of the characteristic equation, the function $\Psi(z)$ has an infinite number of poles, each of which is located in an interval, the end points of which are zeros of $J_{l+\frac{1}{2}}$ and $J_{l+\frac{1}{2}}$. If $z_1, z_2, \ldots$ are the poles, then for the $K$th pole, we have

$$\lim_{\delta\to 0} \Psi(z_K-\delta) = -\infty \qquad \text{and} \qquad \lim_{\delta\to 0} \Psi(z_K+\delta) = +\infty. \tag{18}$$

It follows that $\Psi$ has an infinite number of zeros, that is, there is an infinite number of intervals in which $\Psi$ is positive and then negative, alternately. Also the characteristic equation $\Psi(z) = \alpha^4$ has an infinite number of real solutions, corresponding physically to an infinite number of permissible aperiodic modes for the liquid sphere. In the asymptotic case of $v \to \infty$, we have (Chandrasekhar 1959)

$$\sigma \to \sigma_{l,0}^2 \frac{2l+1}{2(l-1)(2l^2+4l+3)} \frac{R^2}{v}. \tag{19}$$

Another asymptotic case that deserves attention is the case when $z \to \infty$. There, the solutions $z \to z_K$ for $K \to \infty$. Physically, this means that the higher modes of decay can be, to a good approximation, given by $\sigma \simeq z_K^2 v/R^2$ $(K \geqslant 2)$.

The numerical calculations were carried out by means of a computer. We used Newton's method to obtain the roots of the Chandrasekhar characteristic equation. For every order of deformation ($l = 2, 3, 4$), this method was successful for the two lowest aperiodic modes, whereas it fails for higher aperiodic modes because of the proximity to the poles. Accordingly, values of the characteristic equation were calculated in such neighbourhoods, before a method of false position (*regula falsi*) was used.

Previously, numerical results for the lowest two aperiodic modes were already obtained by Chandrasekhar. There, they were given in terms of $\sigma/\sigma_{2,0}$ while the generalized viscosity is given as $\sigma_{2,0}R^2/v$. In the case of a charged liquid drop, the ratio of $\sigma_{l,0}$ to $\sigma_{2,0}$ involves fissility parameters. It may be more convenient to express the solutions in terms of $\sigma/\sigma_{l,0}$ and write the generalized viscosity in terms of $\sigma_{l,0}R^2/v = \alpha^2$. The curves of the numerical solutions are given in figure 1, in which $\sigma/\sigma_{l,0}$ is plotted against $\alpha^2$. For a small value of $\alpha^2$ and a specific order of oscillation, there exist two lowest aperiodic modes. When a critical value $\alpha_{\max}^2(l)$ is reached, complex solutions are obtained for $\alpha^2 > \alpha_{\max}^2$. The values of $\alpha_{\max}^2$ for various $l$ are tabulated in table 1. When expressed in the same units, our results agree with those of Chandrasekhar.

## 6. Periodic motion with damping

For every $l \geqslant 2$, when $\alpha^2 > \alpha_{\max}^2$, the characteristic equation will give complex solutions as well as real solutions. The latter are the solutions of the higher aperiodic modes beyond the first two and are infinite in number. The former complex solutions were obtained by means of the computer and the Newton–Raphson method was used.

We attempted to search numerically for more than one pair of conjugated complex solutions for each value of $\alpha^2 \geqslant \alpha_{\max}^2$, over a relatively large region defined by $0 \leqslant \mathrm{Re}(\sigma/\sigma_{l,0}) \leqslant 4{\cdot}0$ and $0 \leqslant \mathrm{Im}(\sigma/\sigma_{l,0}) \leqslant 2{\cdot}0$. However, it was found that for the various orders investigated, there was only one pair of complex conjugate solutions.

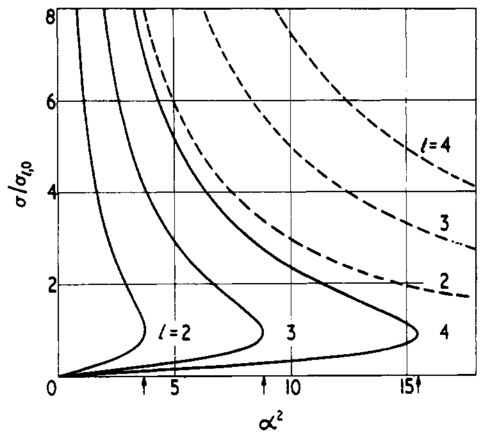


**Figure 1.** The real solutions of the Chandrasekhar characteristic equation for the two lowest aperiodic modes of decay (full curves) and one higher aperiodic mode (broken curves) as a function of $\alpha^2$. They are labelled by the order of the deformations $l = 2, 3$, and 4. The locations of $\alpha^2_{max}$ for various $l$ are indicated by arrows in the figure.

**Table 1.** The values of $\alpha^2_{max}(l)$ and the corresponding $\sigma/\sigma_{l,0}$ for the lowest aperiodic mode. The critical values $\alpha^2_{crit}(l)$ determine the radii $R_{crit}(l)$ below which periodic motion is impossible. As an example the radii $R_{crit}(l)$ for a water drop for various values of $l$ are given.

| $l$ | $\alpha^2_{max}(l)$ | $\dfrac{\sigma}{\sigma_{l,0}}$ (lowest mode) | $R_{crit}$ for water (mm) |
|---|---|---|---|
| 2 | 3·6902 | 0·96799 | $0{\cdot}236 \times 10^{-4}$ |
| 3 | 8·8340 | 0·9260 | $0{\cdot}360 \times 10^{-4}$ |
| 4 | ~15·45 | ~0·90 | $0{\cdot}459 \times 10^{-4}$ |

We plot the real parts of the eigenfrequencies $\mathrm{Re}(\sigma/\sigma_{l,0})$ and their imaginary parts $\mathrm{Im}(\sigma/\sigma_{l,0})$ as functions of $\alpha^2$ in figure 2. As the complex solutions appear in conjugate, we plot only the solutions in the upper complex plane. One observes that as $\alpha^2$ increases, $\mathrm{Im}(\sigma/\sigma_{l,0})$ rises rapidly from 0 towards 1·0. One can see the effect of viscosity on the vibrational frequency, namely, that when periodic solution is still possible, the higher the kinematic viscosity, the smaller is $\alpha^2$ and the lower is the eigenfrequency of vibration. On the other hand, $\mathrm{Re}(\sigma/\sigma_{l,0})$ decreases and approaches 0 with increasing values of $\alpha^2$. Hence, the decaying factor decreases with viscosity as expected. The behaviour of the solution for small viscosity can be understood by considering equation (5). There, as $\nu \to 0$, or equivalently $\alpha^2 \to \infty$, the eigenfrequencies approach asymptotically the values (Chandrasekhar 1959)

$$\frac{\sigma}{\sigma_{l,0}} = \frac{(l-1)(2l+1)}{\alpha^2} \pm \mathrm{i}. \tag{20}$$

One more point can be noted. As $\alpha^2$ varies in the neighbourhood of $\alpha^2_{max}$, the solutions of Chandrasekhar characteristic equations for the aperiodic and the periodic motions vary in a continuous way. This continuity can be seen in figure 2, in which the solutions for the aperiodic modes of decay are also plotted. One observes that the real part of the

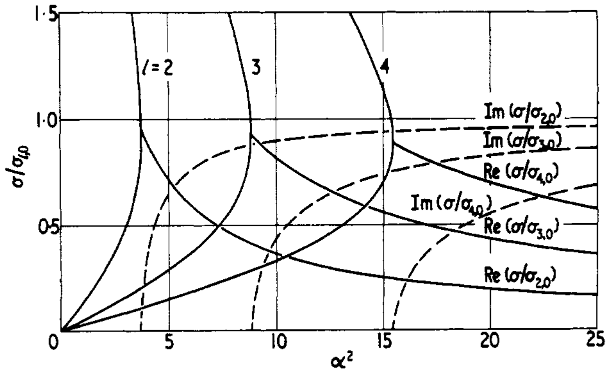


**Figure 2.** The complex solutions $\sigma/\sigma_{l,0}$ of the characteristic equation are plotted against $\alpha^2$. For each $l$, the real part of $\sigma/\sigma_{l,0}$ is shown as a full curve and the imaginary part is shown as a broken curve. The real solution for $\alpha^2 < \alpha^2_{max}$ (full curve) is also included to show the continuity of the real parts as $\alpha^2$ varies across $\alpha^2_{max}$.

eigenfrequencies for the periodic motion joins onto the solutions for the aperiodic motion at $\alpha^2_{max}$. This property was made use of in locating the accurate values of $\alpha^2_{max}(l)$ in the numerical calculations.

## 7. Examples

So far, the viscosity of the fluid has been given a generalized definition so that the same Chandrasekhar characteristic equation applies. It is desirable to see explicitly in a concrete example how the generalized viscosity coefficient $\alpha^2$ and the viscosity coefficient $\eta$ are related. We shall give such dependence in the example of a rain drop. The oscillations of a rain drop are usually caused by the flow of air around them. From the size of the rain drops and their frequencies of oscillation as they fall down, much information may be obtained about the atmospheric conditions (Azad and Latham 1970, Nelson and Gokhale 1972). The relevant coefficients at 20°C are (Weast 1971–1972):

$$\rho_M = 0{\cdot}998203\ \mathrm{g\,cm^{-3}}$$

$$\eta_{20^\circ\mathrm{C}} = 1{\cdot}002\ \mathrm{cp}$$

$$T_{20^\circ\mathrm{C}} = 72{\cdot}75\ \mathrm{dyn\,cm^{-1}}.$$

Assuming a harmonic deformation of order $l$, the eigenfrequencies in the inviscid case are given from equation (4*b*):

$$\sigma^2_{l,0} = l(l-1)(l+2)\frac{T}{\rho_M R^3}.$$

So, the generalized viscosity coefficient $\alpha^2$ can be expressed as

$$\alpha^2 = \frac{\sigma_{l,0}R^2}{\nu} = \left(\frac{l(l-1)(l+2)\rho_M TR}{\eta^2}\right)^{1/2}. \tag{21}$$

In figure 3, we plot $\alpha^2$ and $\sigma_{l,0}$ as functions of radius $R$. As can be seen from the figure, $\alpha^2$ increases with increasing $R$ while $\sigma_{l,0}$ are very large for small values of $R$, but

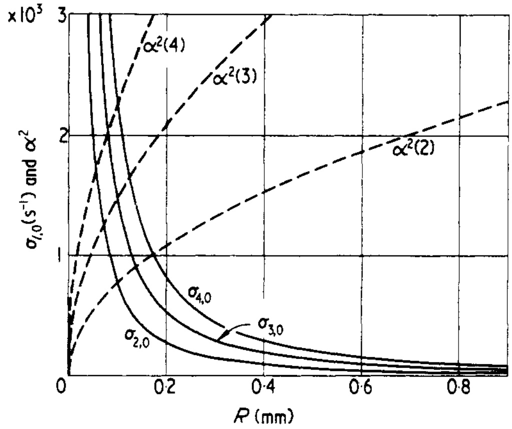


**Figure 3.** The natural eigenfunctions $\sigma_{l,0}$ of a water drop at 20°C (full curve) and the corresponding generalized viscosity coefficient $\alpha^2$ (broken curves) are plotted as functions of radius $R$.

decrease very rapidly towards zero with increasing $R$. For $R$ of the order of 0·1 mm, $\alpha^2$ is of the order of $10^3$. Thus, for water drops of this size, the effect of viscosity is very small.

Corresponding to the initial value $\alpha^2_{\max}$, we have an initial radius $R_{\text{crit}}$ below which there can be only aperiodic modes of motion:

$$R_{\text{crit}} = \frac{(\alpha^2_{\max})^2\eta^2}{l(l-1)(l+2)\rho_M T}. \tag{22}$$

For the water drop, the corresponding values of $\alpha^2_{\max}$ and $R_{\text{crit}}$ for various $l$ are given in table 1. An examination of these initial values reveals the fact that $R_{\text{crit}}$ at $l = 2, 3$, and 4 are rather small from a macroscopic point of view. Atmospheric water drops and those used in the experiments are usually of much larger size. This implies that oscillatory modes with small damping are always present in the motion of moderately large (of the order of 1 mm) water drops which are slightly perturbed. Our result of $R_{\text{crit}}$ for the principal mode $l = 2$ agrees with that of Chandrasekhar (1969) where a factor of $10^4$ missing previously was corrected.

It is interesting to note the sudden change of character of the motion in the neigbourhood of $R_{\text{crit}}$. It is expected that any phenomenon which may be affected by the vibration of the liquid drop, such as the propagation and reflection of sound, heat, and electromagnetic waves in a medium of suspended liquid drops, will exhibit a singular behaviour for drops with radius close to $R_{\text{crit}}$.

For an oscillating liquid sphere, aperiodic motion occurs only when the natural frequency is greater than a certain limit. This is different from the general behaviour of a damped harmonic oscillator with a damping term $\gamma\dot{x}$ for a fixed value of $\gamma$. There, overdamping occurs when the natural frequency $\omega_0$ is lower than a certain limit. The behaviour of an aperiodic motion at high frequency can however be reproduced with a damping term of the form $\gamma\omega_0^2\dot{x}$ or $\gamma\dddot{x}$ in a simple harmonic oscillator.

With the determination of the value of $\alpha^2_{\max}$, one can set an upper limit on the order of magnitude of the coefficient of viscosity for an atomic nucleus. From the fact that vibrational-type spectra have been observed for atomic nuclei (Bohr and Mottelson

1953), one concludes that the nuclear viscosity coefficient for nuclear matter satisfies the following inequalities:

$$\frac{\sigma_{2,0}R^2}{\nu} > 3{\cdot}69,$$

and

$$\frac{\sigma_{3,0}R^2}{\nu} > 8{\cdot}82.$$

For an order of magnitude estimate of the upper limit, one can take $\sigma_{\lambda,0}$ to be the observed energy of the first $\lambda$-multiple states in some vibrational nuclei. In the Cd–Pd region, the first $2^+$ state lies at an energy about 0·5 MeV and the first $3^-$ state about 2 MeV. The inequalities lead to

$$\nu < 0{\cdot}019 \text{ fm } c = 5{\cdot}7 \times 10^{-5} \text{ cm}^2 \text{ s}^{-1}$$

from the quadrupole state and

$$\nu < 0{\cdot}030 \text{ fm } c = 9{\cdot}0 \times 10^{-5} \text{ cm}^2 \text{ s}^{-1}$$

from the octupole state. One concludes that the viscosity coefficient $\nu$ for an atomic nucleus should be less than the order of 0·02 fm $c$.

## 8. Conclusions

Because of the general nature of the Chandrasekhar characteristic equation in being applicable to an incompressible liquid sphere irrespective of the nature of counter-balancing forces, the equation has a wide range of applications for models of astronomical objects, water drops and atomic nuclei. For this reason, the solutions to the Chandrasekhar equation are examined in some detail. Complex solutions to this equation, corresponding to periodic motion of the liquid sphere with damping, are evaluated. In addition, some solutions for higher aperiodic modes are also evaluated.

Of particular interest in future work is the quantization of a charged liquid drop with viscosity, for which the inviscid case has already been treated by Bohr and Mottelson (1953) in the collective model of an atomic nucleus. Understanding of the classical dynamics may be of some help in formulating the problem.

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## Appendix. Vibrational frequency of an inviscid liquid drop with a surface charge and low conductivity

In the extreme case of a liquid with a very low conductivity, the charges endowed on the liquid surface will follow the flow pattern of the liquid instead of redistributing themselves to maintain an equipotential surface. Then, to the first order in $\epsilon$, the surface charge density within an element of solid angle remains constant during the motion; ie,

$$\rho_Q\left(1+\frac{2\epsilon Y_{lm}}{R}\right)R^2\sin\theta\,d\theta\,d\phi = \rho_0 R^2\sin\theta\,d\theta\,d\phi,$$

where $\rho_Q$ is the surface charge density at any instant and $\rho_0$ is the uniform charge density in the equilibrium configuration. Hence we have

$$\rho_Q = \frac{Q}{4\pi R^2}\left(1-\frac{2\epsilon Y_{lm}}{R}\right). \tag{A.1}$$

From the charge distribution, the electric potential at any point in space can be evaluated to be:

$$V(r,\theta,\phi,t) = \begin{cases} kQ\left(\dfrac{1}{r}+\dfrac{l}{2l+1}\dfrac{R^{l-1}}{r^{l+1}}\epsilon Y_{lm}\right) & \text{for exterior points,} \\ kQ\left(\dfrac{1}{R}-\dfrac{l+1}{2l+1}\dfrac{r^l}{R^{l+2}}\epsilon Y_{lm}\right) & \text{for interior points.} \end{cases} \tag{A.2}$$

Then the radial component of the electric field is given as:

$$E_r(r,\theta,\phi,t) = \begin{cases} kQ\left(\dfrac{1}{r^2}+\dfrac{l(l+1)}{2l+1}\dfrac{R^{l-1}}{r^{l+2}}\epsilon Y_{lm}\right) & \text{for exterior points,} \\ kQ\dfrac{l(l+1)}{2l+1}\dfrac{r^{l-1}}{R^{l+2}}\epsilon Y_{lm} & \text{for interior points.} \end{cases} \tag{A.3}$$

The pressure at surface points due to Coulomb repulsion is the mean of the value $\rho_Q E_r$, at interior and exterior points, as they approach the surface. In magnitude, it is given as:

$$p_S(\theta,\phi,t) = \frac{kQ^2}{8\pi R^4}+\frac{l^2-3l-2}{2l+1}\frac{kQ}{4\pi R^5}\epsilon Y_{lm}. \tag{A.4}$$

The averaging of the interior and exterior electric fields is justified as the above result can also be obtained in a careful analysis in which a small but finite thickness of the surface charge distribution is taken into account.

After collecting the contributions to the total deviation in pressure, due to surface tension and Coulomb repulsion, the deviation in pressure at the surface is

$$\left(\frac{\delta p}{\rho_M}\right)_{R+\epsilon Y_{lm}} = \left((l-1)(l+2)\frac{T}{\rho_M R^2}-\frac{l^2-3l-2}{2l+1}\frac{kQ^2}{4\pi R^5}\right)\epsilon Y_{lm}. \tag{A.5}$$

As the presence of electric charges on the surface serves only to impose a constraint on the boundary, so for the case of surface charge, we have from equations (6) and (9):

$$\delta w = \left(\frac{\delta p}{\rho_M}\right)_{R+\epsilon Y_{lm}} = (l+1)\Pi_0 R^l \epsilon Y_{lm}. \tag{A.6}$$

Thus from equations (A.5), (A.6) and (16), it can be easily seen that with a very low conductivity, the eigenfrequencies for the inviscid case are given as:

$$\sigma_{l,0}^2 = \frac{l(l-1)(l+2)}{\rho_M R^3}T-\frac{l(l^2-3l-2)}{2l+1}\frac{kQ^2}{4\pi\rho_M R^6}. \tag{A.7}$$

One notices that for the Coulomb repulsion part, $(l^2-3l-2) > 0$ only when $l \geqslant 4$. This result implies that the presence of the electric charge tends to restore the equilibrium for oscillations with $l < 4$ and becomes disruptive only for $l \geqslant 4$. Thus, when an excessively charged liquid drop of this type undergoes fission due to Coulomb repulsion, the break-up will result in four or more fragments. The critical charge for an $l = 4$ oscillation is:

$$Q_{\text{crit}} = 18\left(\frac{\pi R^3 T}{k}\right)^{1/2}. \tag{A.8}$$

This is 4·5 times larger than the critical charge for the case where the electric charges can redistribute themselves to maintain an equipotential surface. It is of interest to see if this result can be verified experimentally.

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